Latent Action Space for Efficient Planning in Theorem Proving

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- Computationally inefficient when planning
  - Autoregressive generation

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- learn a world model in the latent space for model-based reinforcement learning, and
- learn a latent policy for planning in theorem proving.

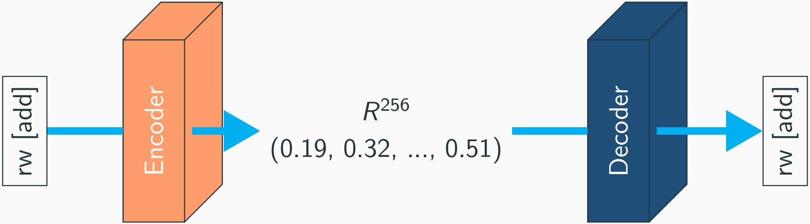
Method

Method — Encoder and Decoder

Learning a high-level representation by embedding the raw action space into a latent action space:

- Encoder_{action}: action space $\rightarrow$ latent action space : $\alpha \sim \texttt{En}(\alpha|a)$.
- Decoder_{action}: latent action space $\rightarrow$ action space : $\hat{a} \sim \texttt{Dn}(\hat{a}|\alpha)$.

Examples — Encoder and Decoder



Method — World Model

Working with the latent action space requires more:

Latent State Space

- State encoder: $z_t \sim \mathtt{En}(z_t|x_t)$
- State decoder: $\hat{x}_t \sim \mathtt{Dn}(\hat{x}_t|z_t)$

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Latent Dynamics

A neural network model to learn the internal dynamics of the theorem proving engine, performing the deduction step of theorem proving.

- Latent transition operator: $\hat{z}_t \sim p(\hat{z}_t | z_{t-1}, \alpha_{t-1})$

Method — Objectives

Reconstruction Loss $\mathcal{L}_{rec}$

```
encoded_action = encoder(raw_actions)
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- $\mathcal{L}_{MSE}$: the mean squared error between the encoded ground truth of next state and the predicted latent state

Method — Objectives

Forward Loss $\mathcal{L}_{forward}$

```
encoded_state = encoder(state)
encoded_action = encoder(action)
predicted_encoded = trans_op(encoded_state, encoded_action)

if using_CE:
    predicted_next_state = decoder(predicted_encoded)
    loss = CE(predicted_next_state, next_state)

elif using_MSE:
    encoded_next_state = encoder(next_state)
    loss = (predicted_encoded - encoded_next_state)^2
```

Method — Objectives

Adding more semantic groundings for the latent state and action space:

$$\mathcal{L} = \mathcal{L}_{rec}(a) + \mathcal{L}_{rec}(x) + \mathcal{L}_{forward}.$$

The latent transition operator allows us to perform efficient planning in the latent space — looking ahead by unrolling the state dynamics for a number of steps.

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- A character-level transformer for the latent representations of both state and action
- An MLP for internal dynamics (*i.e.*, the transition operator) of INT

Experiments — the INT Environment

INT is a simple and lightweight inequality theorem prover suitable for prototyping our approach. An example proof trajectory in INT:

```
  [{
  "state": "to ((b+a)*(a+b))=(((b+a)*(b+a))*1)",
  "action": "@G((b+a)*(a+b))=(((b+a)*(b+a))*~1)$",
  "next_state": "to ((b+a)*(a+b))=((b+a)*(b+a))"
}, {
  "state": "to ((b+a)*(a+b))=((b+a)*(b+a))",
  "action": "@E((b+a)*~(a+b))=((b+a)*(b+a))$",
  "next_state": "to ((a+b)*(b+a))=((b+a)*(b+a))"
}, {
  "state": "to ((a+b)*(b+a))=((b+a)*(b+a))",
  "action": "@A((a+b)*(b+a))=((b+~a)*(b+a))$",
  "next_state": "QED"
}]
```

Experiments — Transformer Configurations

- 256 embedding dimensions
- 8 attention heads
- 1024 hidden dimensions for position-wise feed-forward layers
- a maximum 128 tokens for both training and evaluation examples.

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- How good is the latent policy?
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- Accuracy of predicting "QED"s

Experiments

Table 1: Performance on the test set. $\texttt{BLEU}_{rec-action}$ (resp. $\texttt{BLEU}_{rec-state}$) denotes the BLEU score of the reconstructed actions. $\texttt{BLEU}_{trans}$ denotes the BLEU score of the predicted states by applying the transition operator once. QED accuracy is the percentage of correctly predicted QEDs by applying the transition operator once.

Methods	$\texttt{BLEU}_{rec-action}$	$\texttt{BLEU}_{rec-state}$	$\texttt{BLUE}_{trans}$	QED accuracy (%)
$\mathcal{L}_{CE}$	96.98	94.12	88.23	94.20
$\mathcal{L}_{MSE}$	73.87	69.38	60.18	0

Experiments — Looking ahead

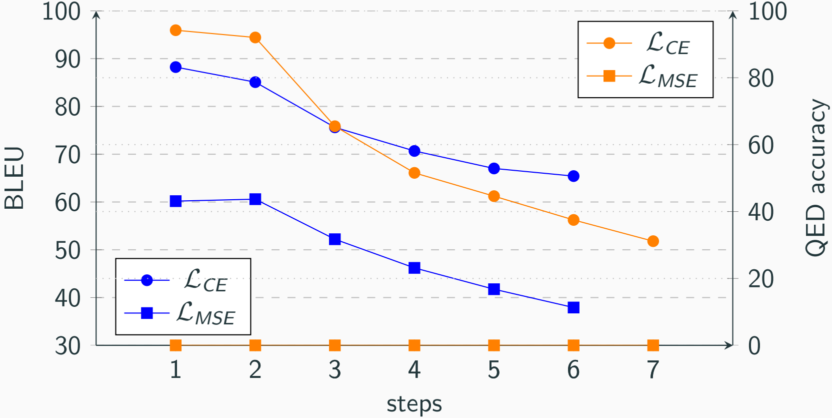

Figure 1: Given a state s, we look ahead n steps by recursively applying the transition operator to s and the subsequent ground truth actions corresponding to s. Note the different scale on right for QED accuracy. Step 7 has a QED accuracy instead of a BLEU score because all target states at step 7 are QEDs.

Experiments — Latent Policy

Table 2: Fix a transition operator learned with $\mathcal{L}_{CE}$. $\texttt{BLEU}_{action}$ denotes the BLEU score of the predicted actions. $\texttt{BLEU}_{next-state}$ denotes the BLEU score of the predicted next states by applying the transition operator once. QED accuracy is the percentage of correctly predicted QEDs by applying the transition operator once to the state and predicted action.

Methods	$\texttt{BLEU}_{action}$	$\texttt{BLEU}_{next-state}$	QED accuracy (%)
$\mathcal{L}_{CE}$	86.73	76.05	18.60
$\mathcal{L}_{MSE}$	85.88	81.82	79.55